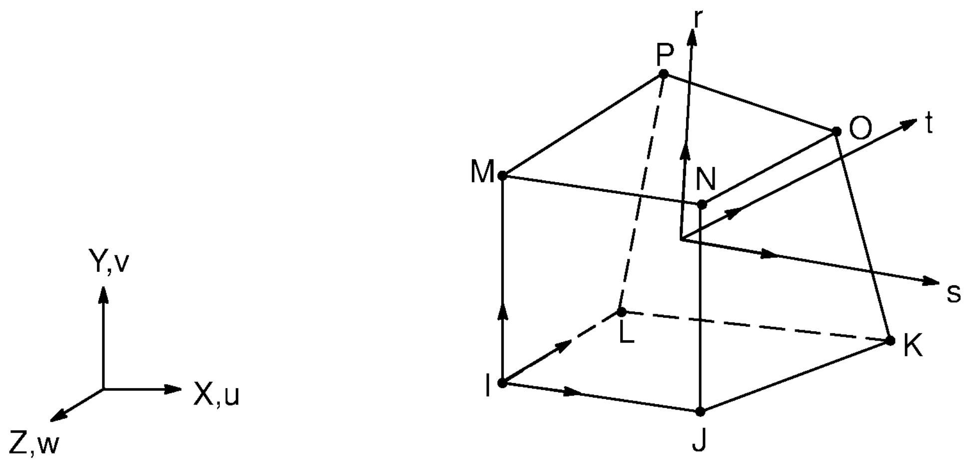
14.86 HYPER86 — 3–D 8–Node Hyperelastic Solid



Matrix or Vector	Shape Functions	Integration Points
Stiffness Matrix	Equations (12.8.18–1), (12.8.18–2) and (12.8.18–3)	2 x 2 x 2 (if KEYOPT(6) = 1, use 1 x 1 x 1 for volumetric terms)
Mass Matrix	Same as stiffness matrix	2 x 2 x 2
Pressure Load Vector	Same as stiffness matrix, specialized to the face	2 x 2

Load Type	Distribution
Element Temperature	Trilinear thru element
Nodal Temperature	Trilinear thru element
Pressure	Bilinear across each face

References: Oden(27), Zienkiewicz(39), Rivlin(89), Kao(90), Mooney(91), and Blatz(92)

14.86.1 Other Applicable Sections

The hyperelastic material models (Mooney–Rivlin and Blatz–Ko) are described in Section 4.5. Section 13.1 describes integration point locations.

14.86.2 Virtual Work Statement

The variational principle employed to derive the incremental stiffness matrix of the hyperelastic finite elements described in this section is the incremental principle of virtual work. Internal and external work as well as their increments are expressed in an equilibrium statement for an element.

$$\delta U + \delta \dot{U} = \delta V + \delta \dot{V} \tag{14.86–1}$$

where:
- δU = internal virtual work
- $\delta \dot{U}$ = increment of internal virtual work
- δV = external virtual work
- $\delta \dot{V}$ = increment of external virtual work

The internal virtual work is expressed as the integral over the volume of the current strain energy density function, W. The external virtual work is the work of the surface pressures over the current surfaces, as well as the work of the nodal point loads. Equation (14.86–1) can be expressed as follows:

$$\int_{vol} \delta W \ d(vol) + \int_{vol} \delta \dot{W} \ d(vol) = \sum_{\substack{\text{element} \\ \text{surfaces}}} \int_S P \ \hat{n}_i \ \delta u_i \ dS + \sum_{\substack{\text{element} \\ \text{nodes}}} F_i^{(n)} \ \delta \Delta_i^{(n)} \tag{14.86–2}$$

where:
- vol = current element volume
- W = strain energy density function per unit current volume
- P = scalar pressure magnitude
- $\hat{n}_i$ = components of the unit normal of the current deformed surface
- $F_i^{(n)}$ = applied nodal forces in the i direction at node n
- δu_i = displacement field variations of the i coordinate
- S = current deformed surface area of elemen
- $\delta \Delta_i^{(n)}$ = variation of nodal displacement in the i direction at node n

14.86.3 Element Matrix Derivation

Equation (14.86–2) is the basic equilibrium relationship used to derive the element stiffness matrix and load vectors. Details of the strain energy density function and its variation with respect to current strain are outlined here.

The strain energy density is a function of the current strain components,

$$W = W\left(C_{ij}\right) \tag{14.86–3}$$

where: C_{ij} = components of the right Cauchy–Green deformation tensor (defined below)

Many forms of this functional dependence are possible. The strain energy density functions available are given in Section 4.5.

Without selecting any particular form of W, expressions for the variation of W and an increment of the variation of W are given as follows:

$$\delta W = \frac{\partial W}{\partial C_{ij}} \; \delta C_{ij} \tag{14.86–4}$$

$$\delta \dot{W} = \frac{\partial^2 W}{\partial C_{ij} \partial C_{kl}} \; \delta C_{ij} \; \dot{C}_{kl} + \frac{\partial W}{\partial C_{ij}} \; \delta \dot{C}_{ij} \tag{14.86–5}$$

The deformation tensor [C] is comprised of the products of the deformation gradients [f]

$$C_{ij} = f_{ki} f_{kj} = \text{component of the Cauchy–Green strain tensor} \tag{14.86–6}$$

$$\delta C_{ij} = \delta f_{ki} f_{kj} + f_{ki} \; \delta f_{kj} \tag{14.86–7}$$

$$\delta \dot{C}_{ij} = \delta f_{ki} \dot{f}_{kj} + \delta f_{kj} \; \dot{f}_{ki} \tag{14.86–8}$$

$$\dot{C}_{ij} = \dot{f}_{ki} f_{kj} + f_{ki} \; \dot{f}_{kj} \tag{14.86–9}$$

where:

$f_{ij} = \frac{\partial x_i}{\partial X_j}$

X_i = undeformed position of a particle in direction i

x_i = $X_i + u_i$ = deformed position of a particle in direction i

u_i = displacement of particle in direction i

Substitution of (14.86–4) through (14.86–9) into (14.86–1) yields an element equilibrium equation in terms of the external loads and internal strains, as shown by:

$$2\int_{vol}\left(\frac{\partial W}{\partial C_{ij}}\, f_{ki}\delta f_{kj} + \frac{\partial W}{\partial C_{ij}}\, \dot{f}_{ki}\delta f_{kj} + 2\frac{\partial^2 W}{\partial C_{ij}\partial C_{kl}}\, f_{mi}\delta f_{nj} f_{mk}\dot{f}_{nl}\right) d(vol) \tag{14.86–10}$$

$$= \sum_{\substack{element\\surfaces}} \int_{S_o} P\, \det\left(\underset{\sim}{f}\right) f_{ij}^{-1}\, \hat{N}_j\, \delta u_i\, dS_o + \sum_{\substack{element\\nodes}} F_i^{(n)}\, \delta\Delta_i^{(n)}$$

where: $\frac{\partial^2 W}{\partial C_{ij}\partial C_{kl}}$ = incremental moduli (fourth order tensor)

$\hat{N}_j$ = components of the normal to the original undeformed surface

δu_i = variation of the i coordinate displacement field

S_o = undeformed surface area over which P acts

It can be shown that there is a common virtual factor to all terms in equation (14.86–10). Converting from tensor to matrix form, equation (14.86–10) becomes:

$$[K_e(u)]\ \{\dot{u}\} = \{F^{pr}\} + \{F^{nd}\} - \{R(u\,)\} \tag{14.86–11}$$

where: $[K_e(u)]$ = current element stiffness matrix

$\{F^{pr}\}$ = total current applied pressures (normal to current surface)

$\{F^{nd}\}$ = total current applied nodal point loads

$\{R(u)\}$ = current Newton–Raphson restoring force vector

$\{\dot{u}\}$ = unknown nodal displacement increments

$\{u\}$ = current total nodal displacements before this solution

Equation (14.86–11) gives the element stiffness equation. The unknown quantity is $\{\dot{u}\}$ while the stiffness matrix and restoring force vector are functions of the current value of displacements {u}, through the deformation gradient [f] and derivations of the strain energy density $\partial W/\partial C_{ij}$ and $\partial^2 W/\partial C_{ij}\partial C_{kl}$.

14.86.4 Reduced Integration on Volumetric Term in Stiffness Matrix

This formulation may produce numerical instability in the nearly incompressible range (Poisson's ratio (ν)   0.5). This can be viewed from the following energy relation:

$$W = \frac{1}{2} \int_{vol} \{\epsilon\}^T \{\sigma\} \, d(vol) \tag{14.86–12}$$

where:
$\{\varepsilon\}$ = strain vector = $\lfloor \epsilon_{xx} \;\; \epsilon_{yy} \;\; \epsilon_{zz} \;\; \epsilon_{xy} \;\; \epsilon_{yz} \;\; \epsilon_{xz} \rfloor^T$
$\{\sigma\}$ = stress vector = $\lfloor \sigma_{xx} \;\; \sigma_{yy} \;\; \sigma_{zz} \;\; \sigma_{xy} \;\; \sigma_{yz} \;\; \sigma_{xz} \rfloor^T$

In the case of isotropic materials, the stress–strain relation in terms of shear and bulk modulus for 3–D stress state is:

$$\{\sigma\} = (G[D_s] + K[D_v]) \{\epsilon\} \tag{14.86–13}$$

where:
$G = \frac{E}{2(1 + \nu)}$ = shear modulus
$K = \frac{E}{3(1 - 2\nu)}$ = bulk modulus

$$[D_s] = \left[\begin{array}{ccc|ccc} \frac{4}{3} & -\frac{2}{3} & -\frac{2}{3} & 0 & 0 & 0 \\ -\frac{2}{3} & \frac{4}{3} & -\frac{2}{3} & 0 & 0 & 0 \\ -\frac{2}{3} & -\frac{2}{3} & \frac{4}{3} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$[D_v] = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

subscript s	=	deviatoric
subscript v	=	volumetric

The bulk modulus used in equation (14.86–13) becomes unbounded as Poisson's ratio, ν, approaches 0.5. Now using equation (14.86–13) in (14.86–12):

$$W = W_s + W_v \qquad (14.86\text{–}14)$$

where:

$$W_s = \frac{1}{2}\int_{vol} \lfloor \epsilon \rfloor \, G[D_s]\{\epsilon\} \, d(vol)$$ = deviatoric (shear) strain energy

$$W_v = \frac{1}{2}\int_{vol} \lfloor \epsilon \rfloor \, K[D_v]\{\epsilon\} \, d(vol)$$ = volumetric strain energy

Now using the derivative given in Section 2.2, the discretized finite element relationship becomes:

$$(\alpha G[K_s] + [K_v]) \, \{u\} = \alpha\left\{F^{nd}\right\} \qquad (14.86\text{–}15)$$

where:

α	=	$\frac{1}{K}$
$[K_s]$	=	stiffness associated with shear energy
$[K_v]$	=	stiffness associated with volumetric energy
$\{u\}$	=	nodal displacements
$\{F^{nd}\}$	=	nodal load vector

As Poisson's ratio (ν) approaches 0.5, α approaches 0.0 so that equation (14.86–15) reduces to:

$$[K_v]\{u\} = \{0\} \qquad (14.86\text{–}16)$$

In equation (14.86–16) if $[K_v]$ is non–singular, only the trivial solution is possible (i.e., {u} = {0}). To enforce a nontrivial solution, $[K_v]$ has to be singular and is achieved by a reduced order integration scheme.

14.86.5 Description of Additional Output Strain Measures

The geometric strain measures output for the hyperelastic element are (1) unit extension and (2) angle change with respect to the global Cartesian system.

The stretch ratio provides the basis for interpretation of the finite strain tensor. The change of length per unit of original length (unit extension) is defined as:

$$\Xi^p = \frac{ds - dS}{dS} = \frac{ds}{dS} - 1 = \Lambda - 1 \qquad (14.86\text{–}17)$$

where: ds = current length
dS = original length
Λ = stretch ratio
Ξ^p = unit extension

Defining the physical components of Lagrangian strain as follows:

$$E^p_{ij} = \frac{E_{ij}}{\sqrt{G_{ii}}\ \sqrt{G_{jj}}} = \frac{\frac{1}{2}\left(C_{ij} - \delta_{ij}\right)}{\sqrt{G_{ii}}\ \sqrt{G_{jj}}} \qquad (14.86\text{–}18)$$

where: E^p_{ij} = Lagrangian strain tensor
C_{ij} = right Cauchy–Green tensor
G_{ij} = metric tensor of reference curvilinear system

The output strains in the directions of the global axes are the unit extensions and are defined as follows:

$$\Xi^p_{ii} = \sqrt{1 - 2E^p_{ii}} - 1 \qquad (14.86\text{–}19)$$

where: Ξ^p_{ii} = unit extension (output quantities UNEXTN (X, Y, Z))
$\sqrt{1 - 2E^p_{ii}}$ = stretch ratio measure

The shear rotations (output quantities ROTANG (XY, YZ, XZ)) are defined as the angle change from the reference configuration. The equation for the angle change $\Delta\phi_{ij}$ is:

$$\Delta\phi_{ij} = \frac{\pi}{2} - \theta_{ij} = \sin^{-1} \frac{2E^p_{ij}}{\left(\sqrt{1 + 2E^p_{ii}}\right)\ \left(\sqrt{1 + 2E^p_{jj}}\right)} \qquad (14.86\text{–}20)$$