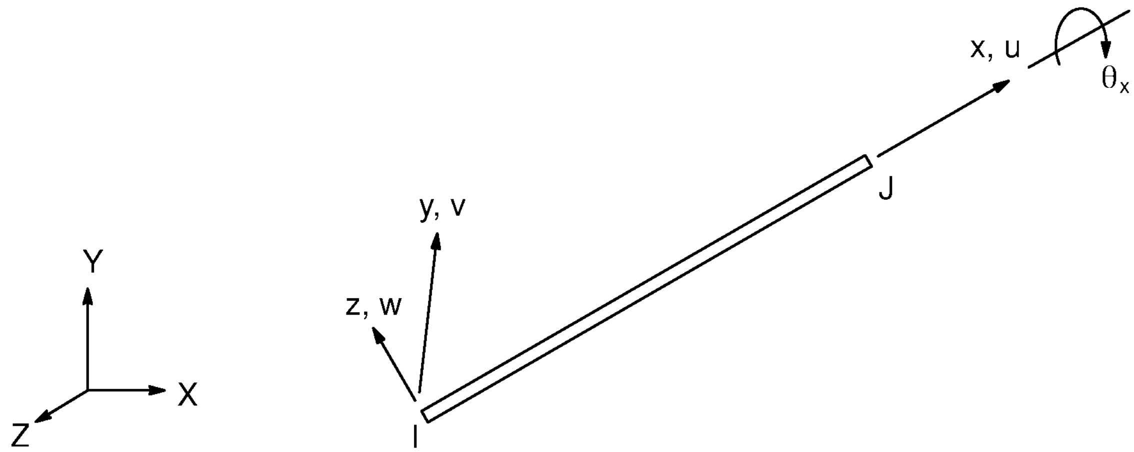
14.4 BEAM4 — 3–D Elastic Beam



Matrix or Vector	Shape Functions	Integration Points
Stiffness Matrix	Equation (12.2.2–1), (12.2.2–2), (12.2.2–3), and (12.2.2–4)	None
Mass Matrix	Same as stiffness matrix	None
Stress Stiffness Matrix	Equation (12.2.2–2) and (12.2.2–3)	None
Pressure Load Vector and Temperatures	Equation (12.2.2–1), (12.2.2–2) and (12.2.2–3)	None

Load Type	Distribution
Element Temperature	Bilinear across cross–section, linear along length
Nodal Temperature	Constant across cross–section, linear along length
Pressure	Linear along length

14.4.1 Stiffness and Mass Matrices

The order of degrees of freedom (DOFs) is shown in Figure 14.4–1.

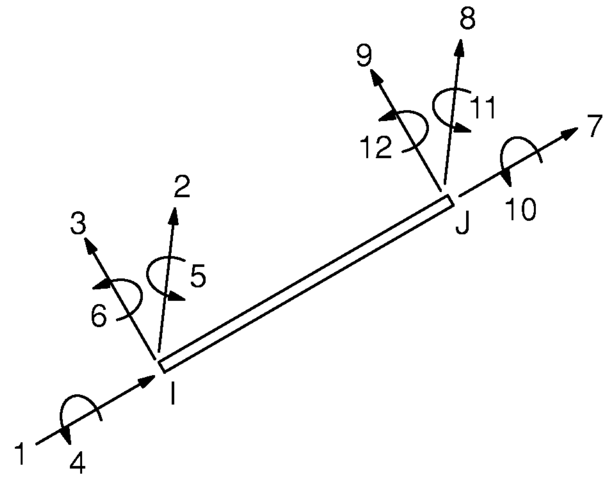


Figure 14.4–1 Order of Degrees of Freedom

The stiffness matrix in element coordinates is (Przemieniecki(28)):

$$[K_\ell] = \begin{bmatrix} AE/L & & & & & & & & & & & \\ 0 & a_z & & & & & & & & \text{Symmetric} & & \\ 0 & 0 & a_y & & & & & & & & & \\ 0 & 0 & 0 & GJ/L & & & & & & & & \\ 0 & 0 & d_y & 0 & e_y & & & & & & & \\ 0 & c_z & 0 & 0 & 0 & e_z & & & & & & \\ -AE/L & 0 & 0 & 0 & 0 & 0 & AE/L & & & & & \\ 0 & b_z & 0 & 0 & 0 & d_z & 0 & a_z & & & & \\ 0 & 0 & b_y & 0 & c_y & 0 & 0 & 0 & a_y & & & \\ 0 & 0 & 0 & -GJ/L & 0 & 0 & 0 & 0 & 0 & GJ/L & & \\ 0 & 0 & d_y & 0 & f_y & 0 & 0 & 0 & c_z & 0 & e_y & \\ 0 & c_z & 0 & 0 & 0 & f_z & 0 & d_y & 0 & 0 & 0 & e_z \end{bmatrix} \quad (14.4\text{–}1)$$

where:

A = cross–section area (input as AREA on **R** command)

E = Young's modulus (input as EX on **MP** command)

L = element length

G = shear modulus (input as GXY on **MP** command)

$$J = \text{torsional moment of inertia} = \begin{cases} J_x \text{ if } I_x = 0 \\ I_x \text{ if } I_x \neq 0 \end{cases}$$

I_x = input as IXX on **RMORE** command

J_x = polar moment of inertia = $I_y + I_z$

z = $a(I_z,\phi_y)$

a_y = $a(I_y,\phi_z)$

b_z = $b(I_z,\phi_y)$

⋮

f_z = $f(I_z,\phi_y)$

f_y = $f(I_y,\phi_z)$

and where:

$$a(I,\phi) = \frac{12EI}{L^3(1+\phi)}$$

$$b(I,\phi) = \frac{-12EI}{L^3(1+\phi)}$$

$$c(I,\phi) = \frac{6EI}{L^2(1+\phi)}$$

$$d(I,\phi) = \frac{-6EI}{L^2(1+\phi)}$$

$$e(I,\phi) = \frac{(4+\phi)EI}{L(1+\phi)}$$

$$f(I,\phi) = \frac{(2-\phi)EI}{L(1+\phi)}$$

and where:

$$\phi_y = \frac{12EI_z}{GA_z^sL^2}$$

$$\phi_z = \frac{12EI_y}{GA_y^sL^2}$$

I_i = moment of inertia normal to direction i (input as Iii on **R** command)

A_i^s = shear area normal to direction i = A/F_i^s

F_i^s = shear coefficient (input as SHEARi on **RMORE** command)

The mass matrix in element coordinates with **LUMPM**,OFF is (Yokoyama(167)):

$$[M_\ell] = M_t \begin{bmatrix} 1/3 & & & & & & & & & & & \\ 0 & A_z & & & & & & & & & \text{Symmetric} & \\ 0 & 0 & A_y & & & & & & & & & \\ 0 & 0 & 0 & J_x/3A & & & & & & & & \\ 0 & 0 & -C_y & 0 & E_y & & & & & & & \\ 0 & C_z & 0 & 0 & 0 & E_z & & & & & & \\ 1/6 & 0 & 0 & 0 & 0 & 0 & 1/3 & & & & & \\ 0 & B_z & 0 & 0 & 0 & D_z & 0 & A_z & & & & \\ 0 & 0 & B_y & 0 & -D_y & 0 & 0 & 0 & A_y & & & \\ 0 & 0 & 0 & J_x/6A & 0 & 0 & 0 & 0 & 0 & J_x/3A & & \\ 0 & 0 & D_y & 0 & F_y & 0 & 0 & 0 & C_y & 0 & E_y & \\ 0 & -D_z & 0 & 0 & 0 & F_z & 0 & -C_z & 0 & 0 & 0 & E_z \end{bmatrix} \quad (14.4\text{–}2)$$

where:

M_t = $(\varrho A+m)L(1-\varepsilon^{in})$

ϱ = density (input as DENS on **MP** command)

m = added mass per unit length (input as ADDMAS on **RMORE** command)

ε^{in} = prestrain (input as ISTRN on **RMORE** command)

A_z = $A(r_z,\phi_y)$

A_y = $A(r_y,\phi_z)$

B_z = $B(r_z,\phi_y)$

⋮

F_z = $F(r_z,\phi_y)$

F_y = $F(r_y,\phi_z)$

and where:

$$A(r,\phi) = \frac{\frac{13}{35} + \frac{7}{10}\phi + \frac{1}{3}\phi^2 + \frac{6}{5}(r/L)^2}{(1+\phi)^2}$$

$$B(r,\phi) = \frac{\frac{9}{70} + \frac{3}{10}\phi + \frac{1}{6}\phi^2 - \frac{6}{5}(r/L)^2}{(1+\phi)^2}$$

$$C(r,\phi) = \frac{\left(\frac{11}{210} + \frac{11}{120}\phi + \frac{1}{24}\phi^2 + \left(\frac{1}{10} - \frac{1}{2}\phi\right)(r/L)^2\right)L}{(1+\phi)^2}$$

$$D(r,\phi) = \frac{\left(\frac{13}{420} + \frac{3}{40}\phi + \frac{1}{24}\phi^2 - \left(\frac{1}{10} - \frac{1}{2}\phi\right)(r/L)^2\right)L}{(1+\phi)^2}$$

$$E(r,\phi) = \frac{\left(\frac{1}{105} + \frac{1}{60}\phi + \frac{1}{120}\phi^2 + \left(\frac{2}{15} + \frac{1}{6}\phi + \frac{1}{3}\phi^2\right)(r/L)^2\right)L^2}{(1+\phi)^2}$$

$$F(r,\phi) = \frac{-\left(\frac{1}{140} + \frac{1}{60}\phi + \frac{1}{120}\phi^2 + \left(\frac{1}{30} + \frac{1}{6}\phi - \frac{1}{6}\phi^2\right)(r/L)^2\right)L^2}{(1+\phi)^2}$$

$$r_y = \sqrt{\frac{I_{yy}}{A}} = \text{radius of gyration}$$

$$r_z = \sqrt{\frac{I_{zz}}{A}} = \text{radius of gyration}$$

The mass matrix in element coordinates with **LUMP**,ON is:

$$[M_\ell] = \frac{M_t}{2}\begin{bmatrix}
1 & & & & & & & & & & & \\
0 & 1 & & & & & & & & & & \\
0 & 0 & 1 & & & & & & & \text{Symmetric} & & \\
0 & 0 & 0 & 0 & & & & & & & & \\
0 & 0 & 0 & 0 & 0 & & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} \quad (14.4\text{–}3)$$

14.4.2 Gyroscopic Damping Matrix

The element gyroscopic damping matrix is the same as for PIPE16, discussed in Section 14.16.

14.4.3 Pressure and Temperature Load Vector

The pressure and temperature load vector are computed in a manner similar to that of BEAM3 (Section 14.3).

14.4.4 Local to Global Conversion

The element coordinates are related to the global coordinates by:

$$\{u_\ell\} = [T_R]\{u\} \tag{14.4–4}$$

where: $\{u_\ell\}$ = vector of displacements in element Cartesian coordinates
$\{u\}$ = vector of displacements in global Cartesian coordinates

$$[T_R] = \begin{bmatrix} T & 0 & 0 & 0 \\ 0 & T & 0 & 0 \\ 0 & 0 & T & 0 \\ 0 & 0 & 0 & T \end{bmatrix}$$

[T] is defined by:

$$[T] = \begin{bmatrix} C_1C_2 & S_1C_2 & S_2 \\ (-C_1S_2S_3 - S_1C_3) & (-S_1S_2S_3 + C_1C_3) & S_3C_2 \\ (-C_1S_2C_3 + S_1S_3) & (-S_1S_2C_3 - C_1S_3) & C_3C_2 \end{bmatrix} \tag{14.4–5}$$

where:

$$S_1 = \begin{cases} \dfrac{Y_2 - Y_1}{L_{xy}} & \text{if } L_{xy} > d \\ 0.0 & \text{if } L_{xy} < d \end{cases}$$

$$S_2 = \frac{Z_2 - Z_1}{L}$$

$$S_3 = \sin(\theta)$$

$$C_1 = \begin{cases} \dfrac{X_2 - X_1}{L_{xy}} & \text{if } L_{xy} > d \\ 1.0 & \text{if } L_{xy} < d \end{cases}$$

$$C_2 = \frac{L_{xy}}{L}$$

C_3 = cos (θ)

X_1, etc. = x coordinate of node 1, etc.

L_{xy} = projection of length onto X–Y plane

d = .0001 L

θ = input as THETA on **R** command

If a third node is given, θ is not used. Rather C_3 and S_3 are defined using:

$\{V_1\}$ = vector from origin to node 1

$\{V_2\}$ = vector from origin to node 2

$\{V_3\}$ = vector from origin to node 3

$\{V_4\}$ = unit vector parallel to global Z axis, unless element is almost parallel to Z axis, in which case it is parallel to the X axis.

Then,

$$\{V_5\} = \{V_3\} - \{V_1\} = \text{vector between nodes I and K} \tag{14.4–6}$$

$$\{V_6\} = \{V_2\} - \{V_1\} = \text{vector along element X axis} \tag{14.4–7}$$

$$\{V_7\} = \{V_6\} \times \{V_4\} \tag{14.4–8}$$

$$\{V_8\} = \{V_6\} \times \{V_5\} \tag{14.4–9}$$

and

$$C_3 = \frac{\{V_7\} \cdot \{V_8\}}{|\{V_7\}| \; |\{V_8\}|} \tag{14.4–10}$$

$$S_3 = \frac{\{V_6\} \cdot (\{V_9\} \times \{V_8\})}{|\{V_6\}| \; |\{V_9\}| \; |\{V_8\}|} \tag{14.4–11}$$

The $\times$ and $\cdot$ refer to vector cross and dot products, respectively. Thus, the element stiffness matrix in global coordinates becomes:

$$[K_e] = [T_R]^T [K_\ell] [T_R] \tag{14.4–12}$$

$$[M_e] = [T_R]^T [M_\ell] [T_R] \tag{14.4–13}$$

$$[S_e] = [T_R]^T [S_\ell] [T_R] \tag{14.4–14}$$

$$\{F_e\} = [T_R]^T \{F_\ell\} \tag{14.4–15}$$

($[S_\ell]$ is defined in Section 3.1).

14.4.5 Stress Calculations

The centroidal stress at end i is:

$$\sigma_i^{dir} = \frac{F_{x,i}}{A} \tag{14.4–16}$$

where:
- σ_i^{dir} = centroidal stress (output quantity SDIR)
- $F_{x,i}$ = axial force (output quantity FX)

The bending stresses are

$$\sigma_{z,i}^{bnd} = \frac{M_{y,i}\, t_z}{2I_y} \tag{14.4–17}$$

$$\sigma_{y,i}^{bnd} = \frac{M_{z,i}\, t_y}{2I_z} \tag{14.4–18}$$

where:
- $\sigma_{z,i}^{bnd}$ = bending stress in element x direction on the element + z side of the beam at end i (output quantity SBZ)
- $\sigma_{y,i}^{bnd}$ = bending stress on the element in element x direction – y side of the beam at end i (output quantity SBY)
- $M_{y,i}$ = moment about the element y axis at end i
- $M_{z,i}$ = moment about the element z axis at end i

t_z = thickness of beam in element z direction (input as TKZ on **R** command)

t_y = thickness of beam in element y direction (input as TKY on **R** command)

The maximum and minimum stresses are:

$$\sigma_i^{max} = \sigma_i^{dir} + \left|\sigma_{z,i}^{bnd}\right| + \left|\sigma_{y,i}^{bnd}\right| \quad (14.4\text{–}19)$$

$$\sigma_i^{min} = \sigma_i^{dir} - \left|\sigma_{z,i}^{bnd}\right| - \left|\sigma_{y,i}^{bnd}\right| \quad (14.4\text{–}20)$$

The presumption has been made that the cross–section is a rectangle, so that the maximum and minimum stresses of the cross–section occur at the corners. If the cross–section is of some other form, such as an ellipse, the user must replace equations (14.4–19) and (14.4–20) with other more appropriate expressions.

For long members, subjected to distributed loading (such as acceleration or pressure), it is possible that the peak stresses occur not at one end or the other, but somewhere in between. If this is of concern, the user should either use more elements or compute the interior stresses outside of the program.