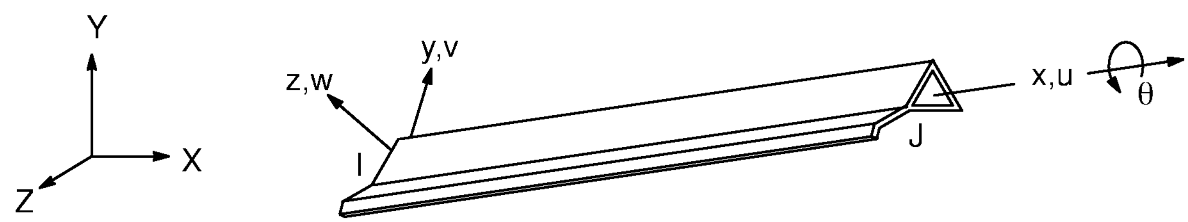
14.24 BEAM24 — 3–D Thin–Walled Beam



Matrix or Vector	Shape Functions	Integration Points
Stiffness Matrix	Equations (12.2.2–1), (12.2.2–2), (12.2.2–3), and (12.2.2–4)	Locations on the cross–section are user defined. No integration points are used along the length for elastic matrix. Same as Newton–Raphson load vector for tangent matrix with plasticity.
Mass Matrix	Same as stress stiffness matrix.	None
Stress Stiffness Matrix	Equations (12.2.2–2) and (12.2.2–3)	None
Thermal Load Vector	Equations (12.2.2–1), (12.2.2–2) and (12.2.2–3)	None
Pressure Load Vector	Equations (12.2.2–2) and (12.2.2–3)	None

Matrix or Vector	Shape Functions	Integration Points
Newton–Raphson Load Vector	Same as thermal load vector	2 along the length and 2 in each segment
Stress Evaluation	Same as thermal load vector	The user defined points on the cross–section are used at each end of the element

Load Type	Distribution
Element Temperature	Bilinear across cross–section and linear along length. See section entitled "Temperature Distribution Across Cross–Section" for more details.
Nodal Temperature	Constant across cross–section, linear along length
Pressure	Linear along length. The pressure is assumed to act along the element x axis.

References: Oden(27), Galambos(13), Kollbrunner(21)

14.24.1 Assumptions and Restrictions

1. The wall thickness is small in comparison to the overall cross–section dimensions (thin–walled theory).
2. The cross–section does not change shape under deformation.
3. St. Venant's theory of torsion governs the torsional behavior. The cross–section is therefore assumed free to warp.
4. Only axial stresses and strains are used in determining the nonlinear material effects. Shear and torsional components are neglected.

14.24.2 Other Applicable Sections

Section 13.1 describes integration point locations. Section 14.4 has an elastic beam element stiffness and mass matrix explicitly written out. Section 14.23 defines the tangent matrix with plasticity, the Newton–Raphson load vector and the stress and strain computation.

14.24.3 Temperature Distribution Across Cross–Section

As stated above, the temperature is assumed to vary bilinearly across the cross–section (as well as along the length). Specifically,

$$T(x,y,z) = \left(T_I + y\left(\frac{\partial T}{\partial y}\right)_I + z\left(\frac{\partial T}{\partial z}\right)_I\right)\left(1 - \frac{x}{L}\right) + \left(T_J + y\left(\frac{\partial T}{\partial y}\right)_J + z\left(\frac{\partial T}{\partial z}\right)_J\right)\frac{x}{L} \quad (14.24\text{–}1)$$

where:
- $T(x,y,z)$ = temperature at integration point located at x, y, z
- x, y, z = location of point in reference coordinate system (coordinate system defined by the nodes
- T_i = temperature at node i (input quantities T1, T4 on **BFE** command)
- $\left(\frac{\partial T}{\partial y}\right), \left(\frac{\partial T}{\partial z}\right)$ = temperature gradients defined below
- L = length

The gradients are:

$$\left(\frac{\partial T}{\partial y}\right)_i = T_{yi} - T_i \quad (14.24\text{–}2)$$

$$\left(\frac{\partial T}{\partial z}\right)_i = T_{zi} - T_i \quad (14.24\text{–}3)$$

where:
- T_{yi} = temperature at one unit from the node i parallel to reference y axis (input quantities T2, T5 on **BFE** command)
- T_{zi} = temperature at one unit from the node i parallel to reference z axis (input quantities T3, T6 on **BFE** command)

14.24.4 Calculation of Cross–Section Section Properties

The cross–section constants are determined by numerical integration, with the integration points (segment points) input by the user. The area of the k^{th} segment (A_k) is:

$$A_k = \ell_k t_k \quad (14.24\text{–}4)$$

where:
- ℓ_k = length of segment k (input indirectly as Y and Z on **R** commands)
- t_k = thickness of segment k (input as TK on **R** commands)

The total cross–section area is therefore

$$A = \sum A_k \tag{14.24–5}$$

where: $\sum$ = implies summation over all the segments

The first moments of area with respect to the reference axes used to input the cross–section are

$$q_y = \frac{1}{2}\sum \left(z_i + z_j\right) A_k \tag{14.24–6}$$

$$q_z = \frac{1}{2}\sum \left(y_i + y_j\right) A_k \tag{14.24–7}$$

where: y_i, z_i = input coordinate locations at beginning of segment k
y_j, z_j = input coordinate locations at end of segment k

The centroidal location with respect to the origin of the reference axes is therefore

$$y_c = q_z/A \tag{14.24–8}$$

$$z_c = q_y/A \tag{14.24–9}$$

where: y_c, z_c = coordinates of the centroid

The moments of inertia about axes parallel to the reference axes but whose origin is at the centroid (y_c, z_c) can be computed by:

$$I_y = \frac{1}{3}\sum \left(\bar{z}_i^2 + \bar{z}_i\bar{z}_j + \bar{z}_j^2\right) A_k \tag{14.24–10}$$

$$I_z = \frac{1}{3}\sum \left(\bar{y}_i^2 + \bar{y}_i\bar{y}_j + \bar{y}_j^2\right) A_k \tag{14.24–11}$$

where: $\bar{y}$ = $y - y_c$
$\bar{z}$ = $z - z_c$

The product moment of inertia is

$$I_{yz} = \frac{1}{3}\sum \left(\bar{y}_i\bar{z}_i + \bar{y}_j\bar{z}_j\right) A_k + \frac{1}{6}\sum \left(\bar{y}_i\bar{z}_j + \bar{y}_j\bar{z}_i\right) A_k \tag{14.24–12}$$

Note that these are simply Simpson's integration rule applied to the standard formulas. The principal moments of inertia are at an angle θ_p with respect to the reference coordinate system Figure 14.24–1, where θ_p (output quantity THETAP) is calculated from:

$$\theta_p = \frac{1}{2} \tan^{-1} \left(\frac{2I_{yz}}{I_z - I_y} \right) \qquad (14.24–13)$$

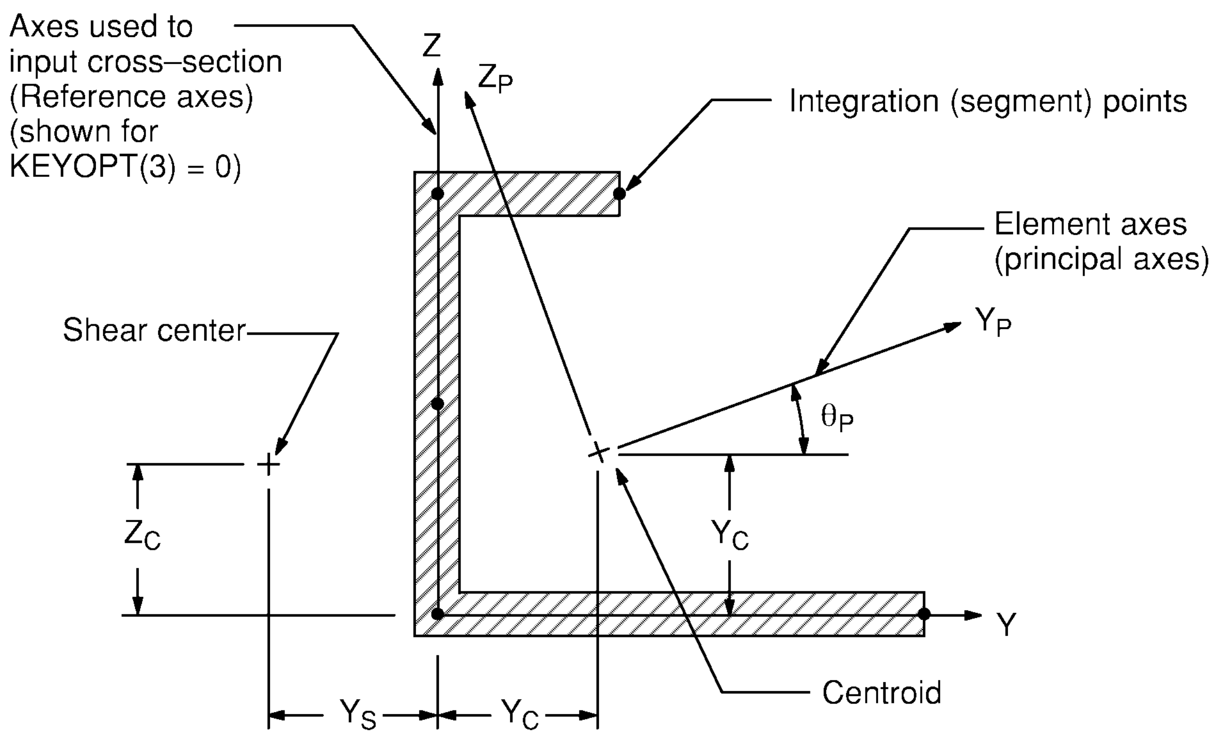


Figure 14.24–1 Cross–Section Input and Principal Axes

The principal moments of inertia with respect to the element coordinate system are therefore:

$$I_{yp} = \frac{1}{2} \left(I_y + I_z \right) + \frac{1}{2} \left(I_y - I_z \right) \cos(2\theta_p) - I_{yz} \sin(2\theta_p) \qquad (14.24–14)$$

and

$$I_{zp} = I_y + I_z - I_{yp} \qquad (14.24–15)$$

where:
I_{yp} = output quantity IYP
I_{zp} = output quantity IZP

The torsional constant for a thin–walled beam of either open or closed (single cell only) cross–section is

$$J = \frac{4 A_o^2}{\sum^{c} \frac{\ell_k}{t_k}} + \frac{1}{3} \sum^{d} \ell_k t_k^3 \qquad (14.24–16)$$

where: J = torsional constant (output quantity J)

A_o = area enclosed by the centerline of the closed part of the cross–section = $\left| \frac{1}{2} \sum^{c} \left(z_i + z_j\right)\left(y_j - y_i\right) \right|$

$\sum^{c}$ = summation over the segments enclosing the area only

$\sum^{d}$ = summation over the remaining segments $\left(\text{not included in } \sum^{c}\right)$

The shear center location with respect to the origin of the reference axes (Figure 14.24–1) is:

$$y_s = y_c + \frac{I_{yz}\, I_{\omega y} - I_z\, I_{\omega z}}{I_{yz}^2 - I_y I_z} \tag{14.24–17}$$

$$z_s = z_c + \frac{I_{yz}\, I_{\omega z} - I_y\, I_{\omega y}}{I_{yz}^2 - I_y I_z} \tag{14.24–18}$$

where: y_s, z_s = output quantity SHEAR CENTER

The sectorial products of inertia used to develop the above expressions are:

$$I_{\omega y} = \frac{1}{3} \sum \left(\omega_i \bar{y}_i + \omega_j \bar{y}_j\right) A_k + \frac{1}{6} \sum \left(\omega_i \bar{y}_j + \omega_j \bar{y}_i\right) A_k \tag{14.24–19}$$

$$I_{\omega z} = \frac{1}{3} \sum \left(\omega_i \bar{z}_i + \omega_j \bar{z}_j\right) A_k + \frac{1}{6} \sum \left(\omega_i \bar{z}_j + \omega_j \bar{z}_i\right) A_k \tag{14.24–20}$$

The sectorial products of inertia are analogous to the moments of inertia, except that one of the coordinates in the definition (such as equation (14.24–12)) is replaced with the sectorial coordinate ω. The sectorial coordinate of a point p on the cross–section is defined as

$$\omega_p = \int_o^s h \, ds \tag{14.24–21}$$

where h is the distance from some reference point (here the centroid) to the cross–section centerline and s is the distance along the centerline from an arbitrary starting point to the point p. h is considered positive when the cross–section is being transversed in the counterclockwise direction with respect to the centroid. Note that the absolute value of the sectorial coordinate is twice the area enclosed by the sector indicated in Figure 14.24–2.

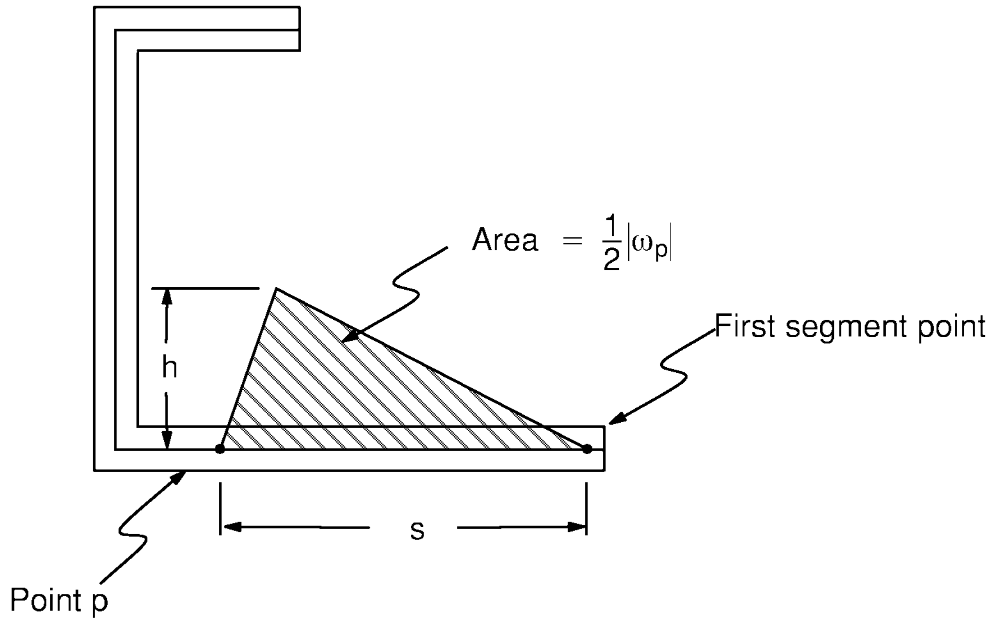


Figure 14.24–2 Definition of Sectorial Coordinate

Equation (14.24–21) can be rewritten using Simpson's integration rule as

$$\omega_p = \sum_1^s \bar{y}_i \left(\bar{z}_j - \bar{z}_i\right) - \bar{z}_i \left(\bar{y}_j - \bar{y}_i\right) \tag{14.24–22}$$

where: $\sum^s$ = summation from the first segment input to the first segment containing point p.

If the segment is part of a closed section or cell, the sectorial coordinate is defined as

$$\omega_p = \sum_1^s \bar{y}_i \left(\bar{z}_j - \bar{z}_i\right) - \bar{z}_i \left(\bar{y}_j - \bar{y}_i\right) - \frac{2\,A_o}{\sum^c \frac{\ell_k}{t_k}} \frac{\ell_k}{t_k} \tag{14.24–23}$$

The warping moment of inertia (output quantity IW) is computed as:

$$I_\omega = \frac{1}{2} \sum \left(\omega_{ni}^2 + \omega_{ni}\omega_{nj} + \omega_{nj}^2\right) A_k \tag{14.24–24}$$

where the normalized sectorial coordinates ω_{ni} and ω_{nj} are defined in general as ω_{np} below. As BEAM24 ignores warping torsion, I_ω is not used in the stiffness formulation but it is calculated and printed for the user's convenience. A normalized sectorial coordinate is defined to be

$$\omega_{np} = \frac{1}{2A} \sum \left(\omega_{oi} + \omega_{oj} \right) A_k - \omega_{op} \qquad (14.24–25)$$

where: ω_{op} = sectorial coordinate with respect to the shear center for integration point p

ω_{op} is defined as with the expressions for the sectorial coordinates equations (14.24–22) and (14.24–23) except that $\bar{y}$ and $\bar{z}$ are replaced by $\tilde{y}$ and $\tilde{z}$. These are defined by:

$$\tilde{y} = y - y_s \qquad (14.24–26)$$

$$\tilde{z} = z - z_s \qquad (14.24–27)$$

Thus, these two equations have been written in terms of the shear center instead of the centroid.

The location of the reference coordinate system affects the line of application of nodal and pressure loadings as well as the member force printout directions. By default, the reference coordinate system is coincident with the y–z coordinate system used to input the cross–section geometry (Figure 14.24–3a). If KEYOPT(3) = 1, the reference coordinate system x axis is coincident with the centroidal line while the reference y and z axes remain parallel to the input y–z axes (Figure 14.24–3b). The shear center and centroidal locations with respect to this coordinate system are

$$\begin{aligned} y_s &= y_{s,o} - y_{c,o} \\ z_s &= z_{s,o} - z_{c,o} \end{aligned} \qquad (14.24–28)$$

and

$$\begin{aligned} y_c &= 0 \\ z_c &= 0 \end{aligned} \qquad (14.24–29)$$

where the subscript o on the shear center and centroid on the right–hand side of equation (14.24–28) refers to definitions with respect to the input coordinate systems in equations (14.24–8), (14.24–9), (14.24–17), and (14.24–18). Likewise, if KEYOPT(3) = 2, the reference x axis is coincident with the shear centerline and the locations of the centroid and shear center are determined to be (Figure 14.24–3c).

$$\begin{aligned} y_c &= y_{c,o} - y_{s,o} \\ z_c &= z_{c,o} - z_{s,o} \end{aligned} \qquad (14.24–30)$$

and

$$\begin{aligned} y_s &= 0 \\ z_s &= 0 \end{aligned} \qquad (14.24\text{–}31)$$

14.24.5 Offset Transformation

The stiffness matrix for a beam element (Section 14.4) is formulated with respect to the element coordinate (principal axis) system for the bending and axial behavior and the shear center for torsional behavior. The stiffness matrix and load vector in this system are $\left[K_\ell\right]$ and $\left\{F_\ell\right\}$. In general, the reference coordinate system in BEAM24 is non–coincident with the element system, hence a transformation between the coordinate systems is necessary. The transformation is composed of a rotational part that accounts for the angle between the reference y and z axes and the element y and z axes (principal axes) and a translational part that accounts for the offsets of the centroid and shear center. The rotational part has the form

$$[R] = \begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix} \qquad (14.24\text{–}32)$$

where:

$$[\lambda] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_p & \sin\theta_p \\ 0 & -\sin\theta_p & \cos\theta_p \end{bmatrix} \qquad (14.24\text{–}33)$$

and θ_p is the angle defined in equation (14.24–13). The translational part is

$$[T] = \begin{bmatrix} I & T_1 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & T_2 \\ 0 & 0 & 0 & I \end{bmatrix} \qquad (14.24\text{–}34)$$

where [I] is the 3 x 3 identity matrix and $[T_i]$ is

$$[T_i] = \begin{bmatrix} 0 & z_c & y_c \\ -z_s & 0 & x_i \\ y_s & -x_i & 0 \end{bmatrix} \quad (14.24\text{–}35)$$

in which y_c, z_c, y_s, and z_s are centroid and shear center locations with respect to the element coordinate system and x_i is the offset in the element x direction for end i. The material to element transformation matrix is then

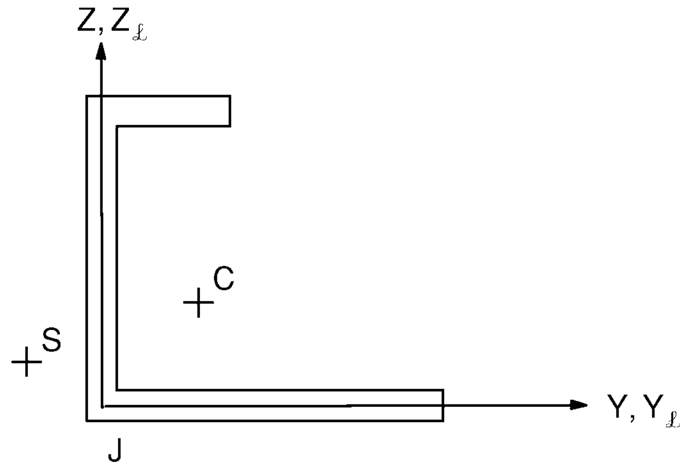
$$[O_f] = [R][T] \quad (14.24\text{–}36)$$

The transformation matrix $[O_f]$ is used to transform the element matrices and load vector from the element to the reference coordinate system

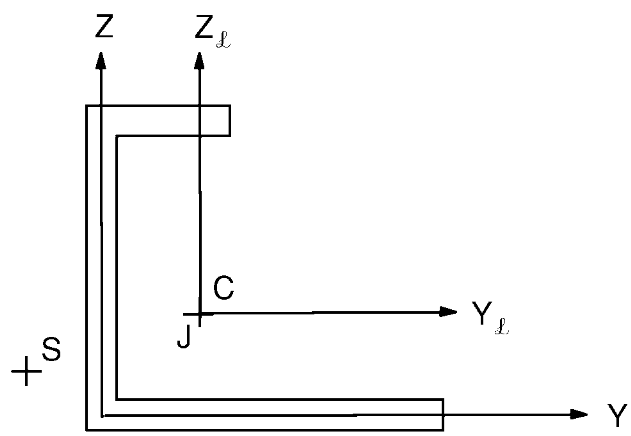
$$[K'_\ell] = [O_f]^T[K_\ell]\,[O_f] \quad (14.24\text{–}37)$$

and

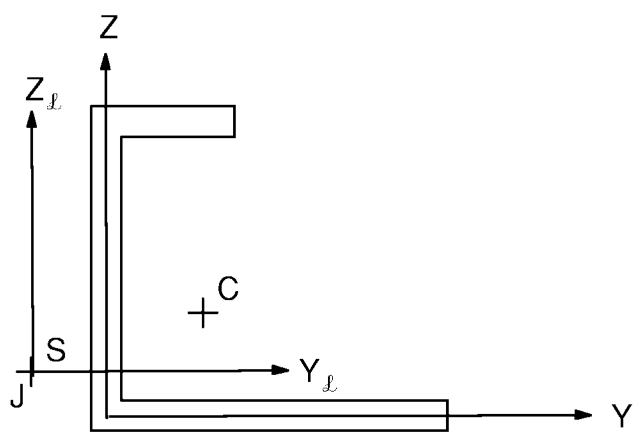
$$\{F'_\ell\} = [O_f]^T\{F_\ell\} \quad (14.24\text{–}38)$$



(a) Default Reference Coordinate System Location (KEYOPT(3)=0)



(b) Reference Coordinate System at Centroid (KEYOPT(3)=1)



(c) Reference Coordinate System at Shear Center (KEYOPT(3)=2)

Figure 14.24–3   Reference Coordinate System

The standard local to global transformation (Section 14.4) can then be used to calculate the element matrices and load vector in the global system:

$$[K_e] = [T_R]^T [K'_\ell] [T_R] \quad (14.24\text{–}39)$$

and

$$\{F_e\} = [T_R]^T \{F'_\ell\} \quad (14.24\text{–}40)$$

The mass and stress stiffening matrices are similarly transformed. The material to element transformation (equation (14.24–37)) for the mass matrix, however, neglects the shear center terms y_s and z_s as the center of mass coincides with the centroid, not the shear center.