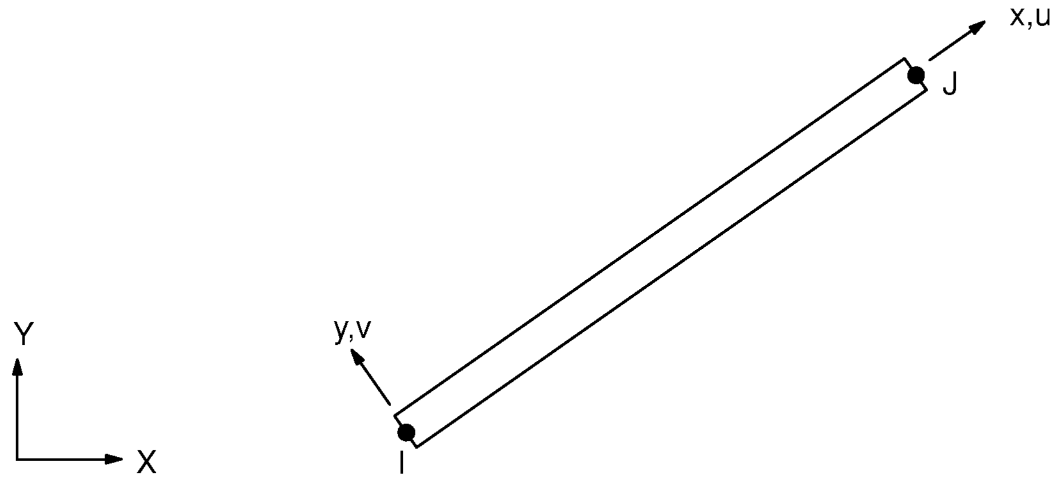
14.23 BEAM23 — 2–D Plastic Beam



Matrix or Vector	Shape Functions	Integration Points
Stiffness Matrix	Equations (12.1.2–1) and (12.1.2–2)	None for elastic matrix. Same as Newton–Raphson load vector for tangent matrix with plasticity
Mass Matrix	Same as stress stiffness matrix	None
Stress Stiffness Matrix	Equation (12.1.2–2)	None
Thermal Load Vector	Same as stress stiffness matrix	None
Pressure Load Vector	Equation (12.1.2–2)	None
Newton–Raphson Load Vector	Same as stiffness matrix	3 along the length and 5 thru the thickness
Stress Evaluation	Same as stiffness matrix	Same as Newton–Raphson load vector

Load Type	Distribution
Element Temperature	Linear thru thickness and along length
Nodal Temperature	Constant thru thickness, linear along length
Pressure	Linear along length

14.23.1 Other Applicable Sections

The complete stiffness and mass matrices for an elastic 2–D beam element (BEAM3) are given in Section 14.3.

14.23.2 Integration Points

There are three sets of integration points along the length of the element, one at each end and one at the middle as shown in Figure 14.23–1.

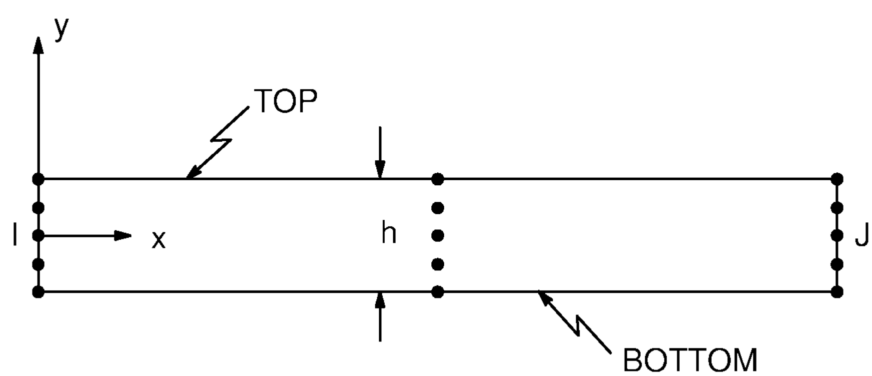


Figure 14.23–1 Integration Point Locations

h is defined as:

h = thickness or height of member (input quantity HEIGHT on **R** command)

The five integration points through the thickness are located at positions y = –0.5 h, –0.3 h, 0.0, 0.3 h, and 0.5 h. Each one of these points has a numerical integration factor associated with it, as well as an effective width, which are different for each type of cross–section. These are derived here in order to explain the procedure used in the element, as well as providing users with a good basis for selecting their own input values for the case of an arbitrary section (KEYOPT(6) = 4).

The criteria used for the element are:

1. The element, when under simple tension or compression, should respond exactly for elastic or plastic situations. That is, the area (A) of the element should be correct.
2. The first moment should be correct. This is nonzero only for unsymmetric cross–sections.
3. The element, when under pure bending, should respond correctly to elastic strains. That is, the (second) moment of inertia (I) of the element should be correct.
4. The third moment should be correct. This is nonzero only for unsymmetric cross–sections.
5. Finally, as is common for numerically integrated cross–sections, the fourth moment of the cross–section (I_4) should be correct.

For symmetrical sections an additional criterion is that symmetry about the centerline of the beam must be maintained. Thus,rather than five independent constants, there are only three. These three constants are sufficient to satisfy the previous three criteria exactly. Some other cases, such as plastic combinations of tension and bending, may not be satisfied exactly, but the discrepancy for actual problems is normally small. For the unsymmetric cross–section case, the user needs to solve five equations, not three. For this case, use of two additional equations representing the first and third moments are recommended. This case is not discussed further here.

The five criteria may be set up in equation form:

$$A = \int_{AREA} dA \tag{14.23–1}$$

$$I_1 = \int_{AREA} y dA \tag{14.23–2}$$

$$I_2 = \int_{AREA} y^2 dA \tag{14.23–3}$$

$$I_3 = \int_{AREA} y^3 dA \tag{14.23–4}$$

$$I_4 = \int_{AREA} y^4 dA \tag{14.23–5}$$

where: dA = differential area
y = distance to centroid

These criteria can be rewritten in terms of the five integration points:

$$A = \sum_{i=1}^{5} H(i)\ L(i)\ h \tag{14.23–6}$$

$$I_1 = \sum_{i=1}^{5} H(i)\ L(i)\ h\ (hP(i)) \tag{14.23–7}$$

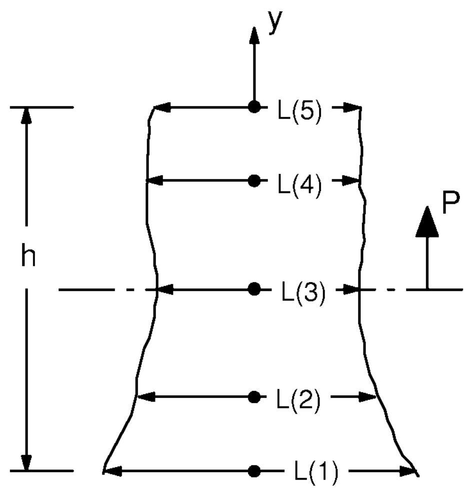
$$I_2 = \sum_{i=1}^{5} H(i)\ L(i)\ h\ \left(hP(i)\right)^2 \tag{14.23–8}$$

$$I_3 = \sum_{i=1}^{5} H(i)\ L(i)\ h\ \left(hP(i)\right)^3 \tag{14.23–9}$$

$$I_4 = \sum_{i=1}^{5} H(i)\ L(i)\ h\ \left(hP(i)\right)^4 \tag{14.23–10}$$

where: H(i) = weighting factor at point i
L(i) = effective width at point i
P(i) = integration point locations in y direction (P(1) = –0.5, P(2) = –0.3, etc.)

The L(i) follows physical reasoning whenever possible as in Figure 14.23–2.



(A(i) = L(i) x h)

Figure 14.23–2 Beam Widths

Starting with the case of a rectangular beam, all values of L(i) are equal to the width of the beam, which is computed from

$$L(i) = \frac{12\ I_{zz}}{h^3} \tag{14.23–11}$$

where: I_{zz} = moment of inertia (input quantity IZZ on **R** command)

Note that the area is not used in the computation of the width. As mentioned before, symmetry may be used to get H(1) = H(5) and H(2) = H(4). Thus, H(1), H(2), and H(3) may be derived by solving the simultaneous equations developed from the above three criteria. These weighting factors are used for all other cross–sections, with the appropriate adjustments made in L(i) based on the same criteria. The results are summarized in Table 14.23–1.

One interesting case to study is that of a rectangular cross–section that has gone completely plastic in bending. The appropriate parameter is the first moment of the area or

$$I_F = \int \left| y \right| \, dA \tag{14.23–12}$$

This results in

$$I_F = \sum_{i=1}^{5} H(i)\ L(i)\ h \left| h\ P(i) \right| \tag{14.23–13}$$

Table 14.23–1 Cross–Sectional Computation Factors

Numerical Integration Point (i)	Location Thru Thickness (P(i))	Numerical Weighting Factor (H(i))	Effective Width (L(i))			
			Rectangular	Pipe	Round Bar	Arbitrary Section
1	–.5	.06250000	$12I_{zz}/h^3$	$8.16445t_p$	$0.25341D_o$	A(–0.5)/h
2	–.3	.28935185	$12I_{zz}/h^3$	$2.64115t_p$	$0.79043D_o$	A(–0.3)/h
3	.0	.29629630	$12I_{zz}/h^3$	$2.00000t_p$	$1.00000D_o$	A(0.0)/h
4	.3	.28935185	$12I_{zz}/h^3$	$2.64115t_p$	$0.79043D_o$	A(0.3)/h
5	.5	.06250000	$12I_{zz}/h^3$	$8.16445t_p$	$0.25341D_o$	A(0.5)/h

where:

P(i) = location, defined as fraction of total thickness from centroid

I_{zz} = moment of inertia (input quantity IZZ on **R** command)

h = thickness (input quantity HEIGHT on **R** command)

t_p = pipe wall thickness (input quantity TKWALL on **R** command)

D_o = outside diameter (input quantity OD on **R** command)

A(i) = effective area based on width at location i (input quantities A(i) on **R** command)

Substituting in the values from Table 14.23–1, the ratio of the theoretical value to the computed value is 18/17, so that an error of about 6% is present for this case.

Note that the input quantities for the arbitrary cross–section (KEYOPT(6) = 4) are h, hL(1)(=A(–50)), hL(2)(=A(–30)), hL(3)(=A(0)), hL(4)(=A(30)), and hL(5)(=A(50)). It is recommended that the user try to satisfy equations (14.23–6) through (14.23–10) using this input option. These equations may be rewritten as:

$$A = 0.06250(A(-50) + A(50)) + 0.28935185(A(-30) + A(30)) + 0.29629630\ A(0) \tag{14.23–14}$$

$$I_1 = (0.0312500(-A(-50) + A(50)) + 0.008680556(-A(-30) + A(30)))h \tag{14.23–15}$$

$$I_2 = (0.01562500(A(-50) + A(50)) + 0.02604170(A(-30) + A(30)))h^2 \tag{14.23–16}$$

$$I_3 = (0.00781250(-A(-50) + A(50)) + 0.00781250(-A(-30) + A(30)))h^3 \tag{14.23–17}$$

$$I_4 = (.00390630(A(-50) + A(50)) + 0.00234375(A(-30) + A(30)))h^4 \tag{14.23–18}$$

Of course, $I_1 = I_3 = 0.0$ for symmetric sections.

Alternative to one of the above five equations, equation (14.23–13) can be used and rewritten as:

$$I_F = (0.031250(A(-50) + A(50)) + 0.08680554(A(-30) + A(30)))h \quad (14.23\text{–}19)$$

Remember that I_2 is taken about the midpoint and that I_{zz} is taken about the centroid. The relationship between these two is:

$$I_{zz} = I_2 - A\, d^2 \quad (14.23\text{–}20)$$

where:

$$d = \frac{I_1}{A} = h \sum_{i=1}^{5} H(i)\ L(i)\ P(I) \Big/ \sum_{i=1}^{5} H(i)\ L(i)$$

$=$ 0.0 for symmetric cross–sections

14.23.3 Tangent Stiffness Matrix for Plasticity

The elastic stiffness, mass, and stress stiffness matrices are the same as those for a 2–D beam element (BEAM3). The tangent stiffness matrix for plasticity, however, is formed by numerical integration. This discussion of the tangent stiffness matrix as well as the Newton–Raphson restoring force of the next subsection has been generalized to include the effects of 3–D plastic beams. The general form of the tangent stiffness matrix for plasticity is:

$$[K_n] = \int_{vol} [B]^T [D_n] [B]\, d(vol) \quad (14.23\text{–}21)$$

where:
$[B]$ = strain–displacement matrix
$[D_n]$ = elasto–plastic stress–strain matrix

This stiffness matrix for a general beam can also be written symbolically as:

$$[K] = \left[K^B\right] + \left[K^S\right] + \left[K^A\right] + \left[K^T\right] \quad (14.23\text{–}22)$$

$[K^B]$ = bending contribution
$[K^S]$ = transverse shear contribution
$[K^A]$ = axial contribution
$[K^T]$ = torsional contribution

where the subscript n has been left off for convenience. As each of these four matrices use only one component of strain at a time, the integrand of equation (14.23–22) can be simplified from $[B]^T[D_n][B]$ to $\{B\}\ D_n \lfloor B \rfloor$. Each of these matrices will be subsequently described in detail.

1. Bending Contribution ($[K^B]$). The strain–displacement matrix for the bending stiffness matrix for bending about the z axis can be written as:

$$\lfloor B^B \rfloor = y \lfloor B_x^B \rfloor \qquad (14.23\text{–}23)$$

where $\lfloor B_x^B \rfloor$ contains the terms of $\lfloor B^B \rfloor$ which are only a function of x (see Narayanaswami and Adelman(129)) :

$$\{B_x^B\} = \frac{1}{L^2 + 12\Phi} \begin{Bmatrix} \frac{12x}{L} - 6 \\ 6x - 4L - \frac{12\Phi}{L} \\ -\left(\frac{12x}{L} - 6\right) \\ 6x - 2L + \frac{12\Phi}{L} \end{Bmatrix} \qquad (14.23\text{–}24)$$

where:

	L	=	beam length
	Φ	=	shear deflection constant (see Section 14.4)

The elasto–plastic stress–strain matrix has only one component relating the axial strain increment to the axial stress increment:

$$D_n = E_T \qquad (14.23\text{–}25)$$

where E_T is the current tangent modulus from the stress–strain curve. Using these definitions equation (14.23–21) reduces to:

$$[K^B] = \int_{vol} \{B_x^B\} E_T y^2 \lfloor B_x^B \rfloor \, d(vol) \qquad (14.23\text{–}26)$$

The numerical integration of equation (14.23–26) can be simplified by writing the integral as:

$$[K^B] = \int_L \{B_x^B\} \left[\int_{area} E_T y^2 \, d(area) \right] \lfloor B_x^B \rfloor \, dx \qquad (14.23\text{–}27)$$

The integration along the length uses a two or three point Gauss rule while the integration through the cross–sectional area of the beam is dependent on the definition of the cross–section. For BEAM23, the integration through the thickness (area) is performed using the 5 point rule described in the previous section. Note that if the tangent modulus is the elastic modulus, $E_T = E$, the integration of equation (14.23–27) yields the exact linear bending stiffness matrix.

The Gaussian integration points along the length of the beam are interior, while the stress evaluation and, therefore, the tangent modulus evaluation is performed at the two ends and the middle of the beam for BEAM23. The value of the tangent modulus used at the integration point in evaluating equation (14.23–27) therefore assumes E_T is linearly distributed between the adjacent stress evaluation points.

2. Transverse Shear Contribution ($[K^S]$). The strain–displacement vector for the shear deflection matrix is (see Narayanaswami and Adelman(129)):

$$\{B^s\} = \frac{6\phi}{L^2 + 12\phi} \left\lfloor -\frac{2}{L} \quad -1 \quad \frac{2}{L} \quad -1 \right\rfloor^T \tag{14.23–28}$$

A plasticity tangent matrix for shear deflection is not required because either the shear strain component is ignored (BEAM23 and BEAM24) or where the shear strain component is computed (PIPE20), the plastic shear deflection is calculated with the initial–stiffness Newton–Raphson approach instead of the tangent stiffness approach. Therefore, since $D_n = G$ (the elastic shear modulus) equation (14.23–21) reduces to:

$$\left[K^S\right] = \int_{vol} \{B^s\}\, G\, \lfloor B^s \rfloor \; d(vol) \tag{14.23–29}$$

Integrating over the shear area explicitly yields:

$$\left[K^S\right] = GA_s \int_L \{B^s\} \lfloor B^s \rfloor \; dx \tag{14.23–30}$$

where A_s is the shear area (see Section 14.3). As $\lfloor B^s \rfloor$ is not a function of x in equation (14.23–28), the integral along the length of the beam in equation (14.23–30) could also be easily performed explicitly. However, it is numerically integrated with the two or three point Gauss rule along with the bending matrix $[K^B]$.

3. Axial Contribution ($[K^A]$). The strain–displacement vector for the axial contribution is:

$$\{B^A\} = \frac{1}{L} \lfloor 1 \quad -1 \rfloor^T \tag{14.23–31}$$

As with the bending matrix, $D_n = E_T$ and equation (14.23–21) becomes:

$$\left[K^{\Lambda}\right] = \int_{vol} \left\{B^{\Lambda}\right\} E_T \left\lfloor B^{\Lambda} \right\rfloor \, d(vol) \tag{14.23–32}$$

which simplifies to:

$$\left[K^{\Lambda}\right] = \int_{L} \left\{B^{\Lambda}\right\} \left[\iint_{area} E_T \, d(area) \right] \left\lfloor B^{\Lambda} \right\rfloor \, dx \tag{14.23–33}$$

The numerical integration is performed using the same scheme as is used for the bending matrix.

4. Torsion Contribution ($[K^T]$). Torsional plasticity (PIPE20 only) is computed using the initial–stiffness Newton–Raphson approach. The elastic torsional matrix (needed only for the 3–D beams) is:

$$\left[K_T\right] = \frac{GJ}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{14.23–34}$$

14.23.4 Newton–Raphson Load Vector

The Newton–Raphson restoring force is:

$$\left\{F_n^{nr}\right\} = \int_{vol} [B]^T [D] \left\{\epsilon_n^{el}\right\} d(vol) \tag{14.23–35}$$

where:
- $[D]$ = elastic stress–strain matrix
- $\left\{\epsilon_n^{el}\right\}$ = elastic strain from the previous iteration

The load vector for a general beam can be written symbolically as:

$$\left\{F^{nr}\right\} = \left\{F_B^{nr}\right\} + \left\{F_S^{nr}\right\} + \left\{F_{\Lambda}^{nr}\right\} + \left\{F_T^{nr}\right\} \tag{14.23–36}$$

where:
- $\left\{F_B^{nr}\right\}$ = bending restoring force
- $\left\{F_S^{nr}\right\}$ = shear deflection restoring force
- $\left\{F_{\Lambda}^{nr}\right\}$ = axial restoring force
- $\left\{F_T^{nr}\right\}$ = torsional restoring force

and where the subscript n has been left off for convenience. Again, as each of the four vectors use only one component of strain at a time, the integrand of equation

(14.23–35) can be simplified from $[B]^T[D]\{\epsilon_n^{el}\}$ to {B} D ϵ_n^{el}. The appropriate {B} vector for each contribution was given in the previous section. The following paragraphs describe D and ϵ_n^{el} for each of the contributing load vectors.

1. Bending Restoring Force $\left(\{F_B^{nr}\}\right)$. For this case, the elasticity matrix has only the axial component of stress and strain, therefore D = E, the elastic modulus. Equation (14.23–35) for the bending load vector is:

$$\{F_B^{nr}\} = E \int_L \{B_x^B\} \left[\iint_{area} y\ \epsilon^{el}\ d(area) \right] dx \tag{14.23–37}$$

The elastic axial strain is computed by:

$$\epsilon^{el} = \phi y + \epsilon^a - \epsilon^{th} - \epsilon^{pl} - \epsilon^{cr} - \epsilon^{sw} \tag{14.23–38}$$

where: ϕ = total curvature (defined below)
ϵ^a = total strain from the axial deformation (defined below)
ϵ^{th} = axial thermal strain
ϵ^{pl} = axial plastic strain
ϵ^{cr} = axial creep strain
ϵ^{sw} = axial swelling strain

The total curvature is:

$$\phi = \lfloor B_x^B \rfloor \{u^B\} \tag{14.23–39}$$

where $\{u^B\}$ is the bending components of the total nodal displacement vector {u}. The total strain from the axial deformation of the beam is:

$$\epsilon_a = \lfloor B^A \rfloor \{u^A\} = \frac{u_{XJ} - u_{XI}}{L} \tag{14.23–40}$$

where: $\{u^A\}$ = axial components for the total nodal displacement vector {u}
u_{XI}, u_{XJ} = axial displacement of nodes I and J

Equation (14.23–37) is integrated numerically using the same scheme outlined in the previous section. Again, since the nonlinear strain evaluation points for the plastic, creep and swelling strains are not at the same location as the integration points along the length of the beam, they are linearly interpolated.

2. Shear Deflection Restoring Force $\left(\left\{F_S^{nr}\right\}\right)$. The shear deflection contribution to the restoring force load vector uses D = G, the elastic shear modulus and the strain vector is simply:

$$\epsilon^{el} = \gamma_S \tag{14.23–41}$$

where γ_S is the average shear strain due to shear forces in the element:

$$\gamma_S = \lfloor B^S \rfloor \left\{u^B\right\} \tag{14.23–42}$$

The load vector is therefore:

$$\left\{F_S^{nr}\right\} = G A_S \gamma_S \int_L \left\{B^S\right\} dx \tag{14.23–43}$$

3. Axial Restoring Force $\left(\left\{F_A^{nr}\right\}\right)$. The axial load vector uses the axial elastic strain defined in equation (14.23–38) for which the load vector integral reduces to:

$$\left\{F_A^{nr}\right\} = E \int_L \left\{B^A\right\} \left[\iint_{area} \epsilon^{el}\ d(area)\right] dx \tag{14.23–44}$$

4. Torsional Restoring Force $\left(\left\{F_T^{nr}\right\}\right)$. The torsional restoring force load vector (needed only for 3–D beams) uses D = G, the elastic shear modulus and the strain vector is:

$$\gamma_T^{el} = \gamma - \gamma^{pl} - \gamma^{cr} \tag{14.23–45}$$

where:

γ_T^{el} = elastic torsional strain

γ = total torsional strain (defined below)

γ^{pl} = plastic shear strain

γ^{cr} = creep shear strain

The total torsional shear strain is defined by:

$$\gamma = \frac{(\theta_{XJ} - \theta_{XI})\ \rho}{L} \tag{14.23–46}$$

where:

θ_{XI}, θ_{XJ} = total torsional rotations from {u} for nodes I, J, respectively.

$\varrho = \sqrt{\left(y^2 + z^2\right)}$ = distance from shear center

The load vector is:

$$\left\{F_T^{nr}\right\} = G \int_L \left\{B^T\right\} \left[\iint_{area} \rho^2 \, \gamma_T^{el} \, d(area) \right] dx \tag{14.23–47}$$

where: $\{B^T\}$ = strain–displacement vector for torsion (same as axial equation (14.23–31)

14.23.5 Stress and Strain Calculation

The modified total axial strain at any point in the beam is given by:

$$\epsilon_n' = \phi^a \, y + \epsilon^a - \epsilon_n^{th} - \epsilon_{n-1}^{pl} - \epsilon_{n-1}^{sw} \tag{14.23–48}$$

where:
- ϕ^a = adjusted total curvature
- ϵ^a = adjusted total strain from the axial deformation
- ϵ_n^{th} = axial thermal strain
- ϵ_{n-1}^{pl} = axial plastic strain from the previous substep
- ϵ_{n-1}^{cr} = axial creep strain from the previous substep
- ϵ_{n-1}^{sw} = axial swelling strain from the previous substep

The total curvature and axial deformation strains are adjusted to account for the applied pressure and acceleration load vector terms. The adjusted curvature is:

$$\phi^a = \phi - \phi^{pa} \tag{14.23–49}$$

where:
- ϕ = $[B^B]\{u^B\}$ = total curvature
- ϕ^{pa} = pressure and acceleration contribution to the curvature

ϕ^{pa} is readily calculated through:

$$\phi^{pa} = \frac{M^{pa}}{EI} \tag{14.23–50}$$

M^{pa} is extracted from the moment terms of the applied load vector (in element coordinates):

$$\left\{F^{pa}\right\} = \left\{F^{pr}\right\} + \left\{F^{ac}\right\} \tag{14.23–51}$$

$\{F^{pr}\}$ is given in Section 14.3 and $\{F^{ac}\}$ is given in Section 17.1. The value used depends on the location of the evaluation point:

$$M^{pa} = \begin{cases} M_I^{pa} & \text{, if evaluation is at end I} \\ \frac{1}{4}\left(M_I^{pa} - M_J^{pa}\right) & \text{, if evaluation is at the middle} \\ M_J^{pa} & \text{, if evaluation is at end J} \end{cases} \qquad (14.23\text{–}52)$$

The adjusted axial deformation strain is:

$$\epsilon^{a} = \epsilon - \epsilon^{pa} \qquad (14.23\text{–}53)$$

where:
ϵ = $[B^A]\{u^A\}$ = total axial deformation strain
ϵ^{pa} = pressure and acceleration contribution to the axial deformation strain

ε^{pa} is computed using:

$$\epsilon^{pa} = \frac{F_x^{pa}}{EA} \qquad (14.23\text{–}54)$$

where F_x^{pa} is calculated in a similar manner to M^{pa}.

From the modified total strain (equation (14.23–48)) the plastic strain increment can be computed (see Section 4.1), leaving the elastic strain as:

$$\epsilon^{el} = \epsilon' - \Delta\epsilon^{pl} \qquad (14.23\text{–}55)$$

where $\Delta\varepsilon^{pl}$ is the plastic strain increment. The stress at this point in the beam is then:

$$\sigma = E\,\epsilon^{el} \qquad (14.23\text{–}56)$$